

Chapter 8 The Z-Transform

- Z-Transform is a powerful mathematical tool for analysis of discrete-time signals and system.
- under simple conditions Z-transform can be converted to represent signals and systems in frequency domain, i.e. Fourier-Transform domain.
- For a given sequence or discrete-time signal $x(n)$ we write the following power series in complex variable $Z = \rho e^{j\theta}$,

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} \quad \underline{(I)}$$

$$X(z) = \mathcal{Z} \{x(n)\} \quad \text{or} \quad x(n) \xrightarrow{\mathcal{Z}} X(z)$$

where $X(z)$ is a Continuous complex function of complex variable z .

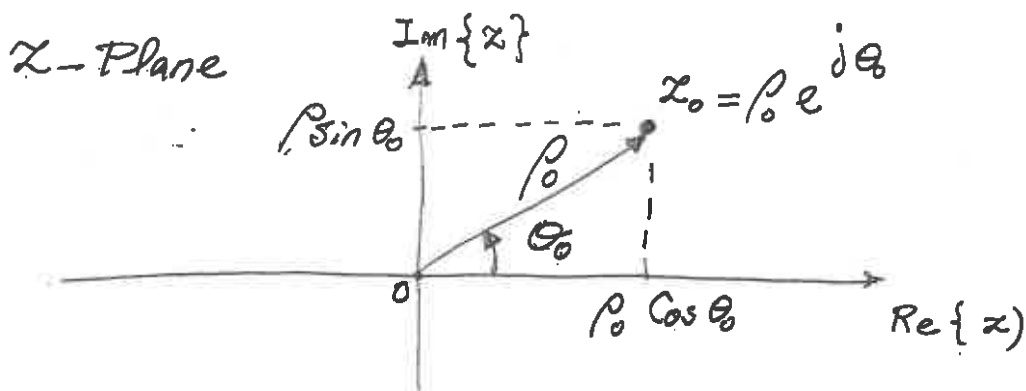
Thus $X(z) = |X(z)| e^{j\angle X(z)}$ where magnitude

$|X(z)|$ and phase $\angle X(z) = \angle X(z)$ are Continuous functions of z .

- $X(z)$ given by the power series in (I) is the $\mathcal{Z}$ -Transform of $x(n)$ if there is a set of complex values z_0 for which (I) converges and exists, that is if

$$|X(z_0)| < \infty, \text{ for some complex or real values of } z = z_0.$$

Note 1: The set of points z_0 for which $X(z_0)$ exists is called the region of Convergence (ROC_x) of $X(z)$ for a given $x(n)$.



Note 2: If $x(n)$ has a $\mathcal{Z}$ -Transform $X(z)$, then $X(z)$ must be defined together with ROC_x . Then, the $\mathcal{Z}$ -Transform is a one-to-one transformation, i.e.,

$$x(n) \xleftrightarrow{\mathcal{Z}} X(z) = \mathcal{Z}\{x(n)\},$$

$$\text{and } ROC_x = r_1 < |z| < r_2,$$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

Examples and definitions: Finding $X(z)$ and ROC_X for $x(n)$.

Example 1:

Find Z -Transform, $X(z)$ and ROC_X , for the finite length (finite duration) signal

$$x(n) = \{ \underset{n=-2}{2}, \underset{n=-1}{4}, \underset{\substack{\uparrow \\ n=0}}{5}, \underset{n=1}{7}, \underset{n=2}{0}, \underset{n=3}{1} \} \quad , \text{ Non-causal signal } x(n).$$

i.e.

$$x(n) = \{ \underset{\substack{\uparrow \\ n=0}}{x(-2)} \quad x(-1) \quad x(0) \quad x(1) \quad x(2) \quad x(3) \}$$

Now:

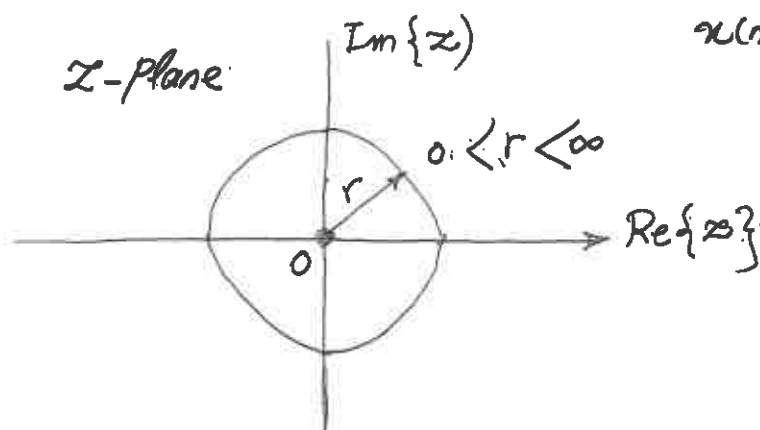
$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = 2z^{+2} + 4z^{+1} + 5 + 7z^{-1} + z^{-3}$$

Only $z = \infty$ and $z = 0$ make $|X(z)| \rightarrow \infty$.

The $X(z)$ is finite and exists everywhere in Z -plane except at $z = +\infty$ and $z = 0$.

Thus

$$ROC_X : r_1 = 0 < |z| < r_2 = \infty \quad \text{for non-causal } x(n).$$



Example 2: Find $X(z)$ and ROC_x for

$$x(n) = \{0, 0, 5, 7, 0, 1\},$$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

Causal signal
 $x(n)$

$$X(z) = \sum_{n=0}^{\infty} x(n) z^{-n} = 5 + 7z^{-1} + z^{-3}$$

$X(z)$ exists everywhere in z -plane except $z=0$.

Thus

$$ROC_x: r_1 = 0 < |z| \text{ for causal } x(n)$$

Example 3: Find $X(z)$ and ROC_x for

$$x(n) = \{2, 4, 0, 0, 0, 0\},$$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

Anti-causal
signal

$$X(z) = \sum_{n=-2}^{-1} x(n) z^{-n} = 2z^{+2} + 4z^{+1}$$

$X(z)$ exists everywhere in z -plane except at $z=\infty$.

$$\text{Thus } ROC_x: 0 \leq |z| < r_2 = \infty$$

for Anti-causal $x(n)$.

Example 4: Find $X(z)$ and ROC_X for the following $x(n)$ signals

- a) $x(n) = A \delta(n)$ causal $x(n)$
- b) $x(n) = A \delta(n-k)$ causal $x(n)$
- c) $x(n) = A \delta(n+k)$ Anti-causal $x(n)$
- d) $x(n) = a^{+n} u(n)$ causal $x(n)$
- e) $x(n) = -a^{+n} u(-n-1)$ Anti-causal, $x(n)$
- f) $x(n) = a^n u(n) - b^{+n} u(-n-1)$ Non-causal $x(n)$,
 $|b| > |a|$.

Solutions: For $X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$:

a) $X(z) = A$, ROC_X for all z in z -plane, $-\infty \leq |z| \leq \infty$

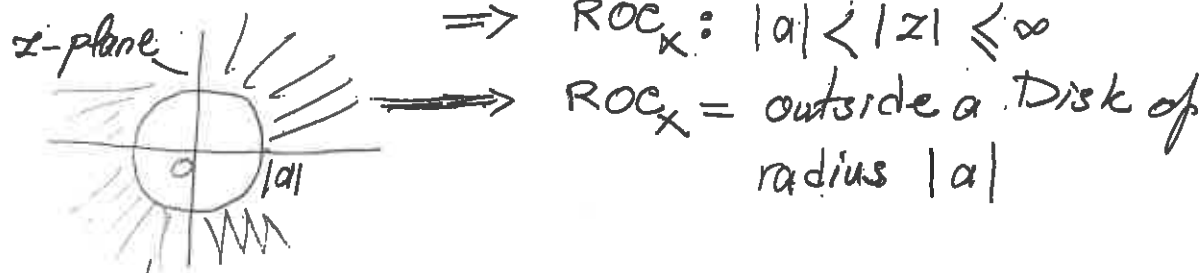
b) $X(z) = A z^{-k}$ $ROC_X: \infty \geq |z| > 0$

c) $X(z) = A z^{+k}$ " $0 \leq |z| < \infty$

d) $X(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \frac{1 - (a z^{-1})^{\infty}}{1 - a z^{-1}} \Big|_{|a z^{-1}| < 1} = \frac{1}{1 - a z^{-1}}$

thus $ROC_X: |a z^{-1}| < 1$

$\Rightarrow ROC_X: |a| < |z| \leq \infty$



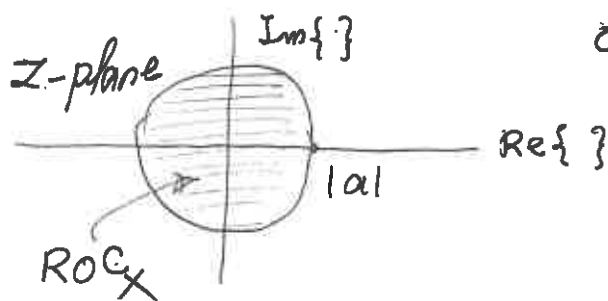
e) $x(n) = -a^n u(-n-1)$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = - \sum_{n=-\infty}^{\infty} a^n u(-n-1) z^{-n}$$

$$X(z) = - \sum_{n=-\infty}^{-1} (\bar{a}^{-1} z)^{-n} = - \sum_{n=1}^{\infty} (\bar{a}^{-1} z)^n$$

$$X(z) = - \frac{\bar{a}^{-1} z - (\bar{a}^{-1} z)^{\infty}}{1 - \bar{a}^{-1} z} = \frac{-\bar{a}^{-1} z}{1 - \bar{a}^{-1} z} \quad | \bar{a}^{-1} z | < 1$$

$$X(z) = \frac{1}{1 - a z^{-1}} \quad \text{ROC}_X: | \bar{a}^{-1} z | < 1$$



$$0 \leq |z| < |a|$$

f) $x(n) = a^n u(n) - b^n u(-n-1)$

$$X(z) = \frac{1}{1 - a z^{-1}} + \frac{1}{1 - b z^{-1}}$$

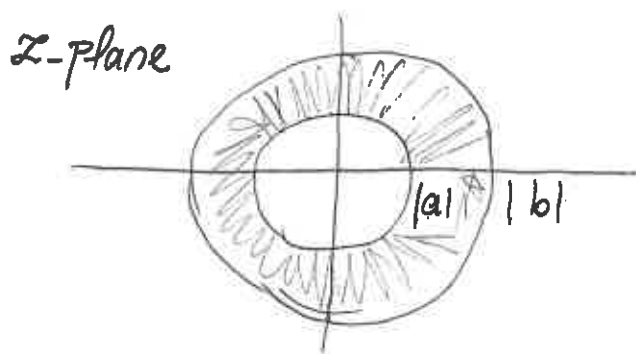
$$\text{ROC}_1: |a| < |z| \leq \infty$$

$$\text{ROC}_2: 0 \leq |z| < |b|$$

Thus ROC_X is INTERSECTION OF ROC_1 and ROC_2 .

$$ROC_X = ROC_1 \cap ROC_2 : |a| < |z| < |b|$$

$$\text{where } |b| > |a|$$



Conclusion

- For Causal $x(n)$, ROC_X is outside a disk of radius r_1 ;
 $r_1 < |z| \leq \infty$.
- For Anti-Causal $x(n)$, ROC_X is inside a disk of radius r_2 ,
 $0 \leq |z| < r_2$
- For Non-causal $x(n)$, ROC_X is a ring in z-plane ;
 $r_1 < |z| < r_2$
- For $x(n) = x_1(n) + x_2(n) + \dots + x_L(n)$ then
 $X(z) = X_1(z) + X_2(z) + \dots + X_L(z)$
 $ROC_X = ROC_{X_1} \cap ROC_{X_2} \cap \dots \cap ROC_{X_L}$

- Any $x(n)$ where ROC_x is empty (i.e. does not exist)

the $x(n)$ DOES NOT have Z-Transform, $X(z)$.

→ Example: periodic signal $x(n) = x(n \pm N)$.

Example: Find Z-transform and ROC_x for $x(n) = u(n)$.

For $v(n) = a^n u(n) \longleftrightarrow V(z) = \frac{1}{1 - az^{-1}}$, $ROC_v: |z| > |a|$.

Now let $a=1$, then

$x(n) = u(n) \longleftrightarrow X(z) = \frac{1}{1 - z^{-1}}$, $ROC_x: |z| > 1$.

Properties of the Z-Transform

$$x(n) \xleftrightarrow{\mathcal{Z}} X(z) \quad \text{ROC}_X: r_1 < |z| < r_2$$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

(I) Linearity (superposition property):

$$\text{If } x_1(n) \longleftrightarrow X_1(z), \text{ ROC}_{X_1}$$

$$x_2(n) \longleftrightarrow X_2(z), \text{ ROC}_{X_2}$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$x_L(n) \longleftrightarrow X_L(z), \text{ ROC}_{X_L}$$

Then

$$x(n) = \sum_{l=1}^L a_l x_l(n) \longleftrightarrow X(z) \quad \text{where}$$

$$\text{ROC}_X = \bigcap_{l=1}^L \text{ROC}_{X_l} \neq \text{empty}$$

(II) Time Shifting:

$$\text{If } x(n) \longleftrightarrow X(z), \text{ ROC}_X: r_1 < |z| < r_2$$

$$\text{then } v(n) = x(n \mp k) \longleftrightarrow V(z) = z^{\mp k} X(z)$$

$$\text{ROC}_V = \text{ROC}_X$$

(III) Scaling in the Z-Domain:

$$\text{If } x(n) \longleftrightarrow X(z), \text{ ROC}_x: r_1 < |z| < r_2$$

$$\text{then } v(n) = a^n x(n) \longleftrightarrow V(z) = X(a^{-1} z)$$

$$\text{for any real or complex Constant } a. \quad \text{ROC}_v: r_1 |a| < |z| < r_2 |a|$$

(IV) Time Reversal (Folding in time-domain):

$$\text{If } x(n) \longleftrightarrow X(z), \quad r_1 < |z| < r_2 \text{ as } \text{ROC}_x$$

$$\text{then } v(n) = x(-n) \longleftrightarrow V(z) = X(\bar{z}^{-1}) \text{ and}$$

$$\text{ROC}_v: \frac{1}{r_2} < |z| < \frac{1}{r_1}$$

(V) Differentiation in Z-Domain:

$$\text{If } x(n) \longleftrightarrow X(z), \quad r_1 < |z| < r_2 \text{ as } \text{ROC}_x$$

$$\text{then } v(n) = n x(n) \longleftrightarrow V(z) = -z \frac{dX(z)}{dz}$$

$$\text{ROC}_v = \text{ROC}_x$$

(VI) Convolution in Time-Domain:

$$\text{If } x_1(n) \longleftrightarrow X_1(z), \text{ ROC}_{x_1}$$

$$x_2(n) \longleftrightarrow X_2(z), \text{ ROC}_{x_2}$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$x_L(n) \longleftrightarrow X_L(z), \text{ ROC}_{x_L}$$

Then

$$v(n) = x_1(n) * x_2(n) * \dots * x_L(n) \longleftrightarrow V(z) = X_1(z) X_2(z) \dots X_L(z)$$

$$\text{ROC}_v = \bigcap_{l=1}^L \text{ROC}_{x_l}, \text{ if exists.}$$

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(VII) Multiplication in Time-Domain:

If $x_1(n) \longleftrightarrow X_1(z)$, ROC_{X_1} $r_{1l} < |z| < r_{1u}$
and $x_2(n) \longleftrightarrow X_2(z)$, ROC_{X_2} $r_{2l} < |z| < r_{2u}$
then

$$v(n) = x_1(n) x_2(n) \longleftrightarrow V(z) = \frac{1}{2\pi j} \oint_C X_1(\alpha) X_2\left(\frac{z}{\alpha}\right) \alpha^{-1} d\alpha$$

- $V(z)$ is linear convolution of two complex-valued continuous functions of complex-valued variable z .
- C is a closed contour that encloses origin ($z=0$) and lies within the ROC common to both $X_1(\alpha)$ and $X_2(\frac{z}{\alpha})$. α is complex-valued variable.

And ROC_V is $r_{1l} r_{2l} < |z| < r_{1u} r_{2u}$.

Note: $x_1(n)$ and $x_2(n)$ are assumed real-valued signals.

If $x_1(n)$ and $x_2(n)$ complex-valued then

$$v(n) = x_1(n) x_2^*(n) \longleftrightarrow V(z) = \frac{1}{2\pi j} \oint_C X_1(\alpha) X_2^*\left(\frac{z^*}{\alpha^*}\right) \alpha^{-1} d\alpha$$

(VIII) Parseval's Relation:

If $x_1(n) \longleftrightarrow X_1(z)$, $r_{1l} < |z| < r_{1u} \Rightarrow \text{ROC}_{X_1}$
 and $x_2(n) \longleftrightarrow X_2(z)$, $r_{2l} < |z| < r_{2u} \Rightarrow \text{ROC}_{X_2}$

then

$$G = \sum_{n=-\infty}^{\infty} x_1(n)x_2(n) = \frac{1}{2\pi j} \oint_C X_1(\alpha) X_2\left(\frac{1}{\alpha}\right) \alpha^{-1} d\alpha$$

where $r_{1l} r_{2l} < 1 < r_{1u} r_{2u}$

that is: Intersection of ROC_{X_1} and ROC_{X_2}
 include unit circle (i.e. a circle
 of radius one in z -plane).

(IX) The initial value Theorem:

If $x(n)$ is causal and $x(n) \longleftrightarrow X(z)$,
 ROC_x

Then $x(0) = \lim_{z \rightarrow \infty} X(z)$.

Extra Note:

- Correlation signal between $x(n)$ and $y(n)$ is defined as

$$r_{xy}(l) = \sum_{n=-\infty}^{\infty} x(n)y(n-l) \quad \text{for } -\infty \leq l \leq \infty$$

and is linear convolution

$$r_{xy}(l) = x(l) * y(-l)$$

Thus

If $x(n) \longleftrightarrow X(z)$, $ROC_X: r_{1l} < |z| < r_{1u}$
 and $y(n) \longleftrightarrow Y(z)$, $ROC_Y: r_{2l} < |z| < r_{2u}$
 then

$$r_{xy}(l) \longleftrightarrow R_{xy}(z) = X(z)Y(z^{-1})$$

and ROC_{xy} : Intersection of $ROC_X: r_{1l} < |z| < r_{1u}$

and $ROC_{Y(z^{-1})}: \frac{1}{r_{2u}} < |z| < \frac{1}{r_{2l}}$,
 if it exists.

Some Examples using properties of Z-Transform:

Example 1: Find $V(z)$ for $v(n) = n \cdot u(n)$

$$u(n) \longleftrightarrow U(z) = \frac{1}{1-z^{-1}} \quad |z| > 1.$$

$$\text{and } v(n) = n u(n) \longleftrightarrow V(z) \quad |z| > 1$$

$$\text{where } V(z) = -z \frac{dU(z)}{dz} = -z \left(\frac{-z^{-2}}{(1-z^{-1})^2} \right)$$

$$V(z) = \frac{z^{-1}}{(1-z^{-1})^2}.$$

Note For $g(n) = n^2 u(n) = n v(n)$, we have

$$G(z) = -z \frac{dV(z)}{dz}.$$

Example 2: Find $V(z)$ for $v(n) = n a^n u(n)$.

$$x(n) = a^n u(n) \longleftrightarrow X(z) = \frac{1}{1-a z^{-1}} \quad |z| > |a|$$

then

$$v(n) = n a^n u(n) \longleftrightarrow V(z) = -z \frac{dX(z)}{dz}$$

$$V(z) = \frac{a z^{-1}}{(1-a z^{-1})^2}$$

$$\text{ROC}_V: |z| > |a|$$

Example 3: Find $X(z)$ for $x(n) = a^n [\cos(\omega_0 n)] u(n)$.

Let $y(n) = \cos(\omega_0 n) u(n) = \frac{1}{2} (e^{j\omega_0})^n u(n) + \frac{1}{2} (\bar{e}^{j\omega_0})^n u(n)$

Since $b^n u(n) \xleftrightarrow{Z} \frac{1}{1 - b\bar{z}^{-1}} \quad |z| > |b|$

then

$$\frac{1}{2} (e^{j\omega_0})^n u(n) \xleftrightarrow{Z} \frac{1/2}{1 - e^{j\omega_0} \bar{z}^{-1}} \quad |z| > 1$$

$$\frac{1}{2} (\bar{e}^{j\omega_0})^n u(n) \xleftrightarrow{Z} \frac{1/2}{1 - \bar{e}^{j\omega_0} \bar{z}^{-1}} \quad |z| > 1$$

then

$$Y(z) = \frac{1}{2} \left\{ \frac{1}{1 - e^{j\omega_0} \bar{z}^{-1}} + \frac{1}{1 - \bar{e}^{j\omega_0} \bar{z}^{-1}} \right\} \quad |z| > 1$$

$$Y(z) = \frac{1 - \cos(\omega_0) \bar{z}^{-1}}{1 - 2 \cos(\omega_0) \bar{z}^{-1} + \bar{z}^{-2}} \quad |z| > 1$$

then

$$x(n) = a^n y(n) \xleftrightarrow{Z} X(z) = Y(a^{-1} z)$$

$$|z| > |a|$$

where

$$X(z) = \frac{1 - a \cos(\omega_0) \bar{z}^{-1}}{1 - 2 a \cos(\omega_0) \bar{z}^{-1} + a^2 \bar{z}^{-2}}$$

$$|z| > |a|$$

Rational Z-Transforms

Definition:

$$X(z) = \frac{P_M(z)}{D_N} = \frac{b_0 + b_1 z^{-1} + \dots + b_M z^{-M}}{1 + a_1 z^{-1} + a_2 z^{-2} + \dots + a_N z^{-N}} = \frac{\sum_{m=0}^M b_m z^{-m}}{1 + \sum_{k=1}^N a_k z^{-k}}$$

or

$$X(z) = \left(b_0 z^{N-M} \right) \frac{z^M + \hat{b}_1 z^{M-1} + \hat{b}_2 z^{M-2} + \dots + \hat{b}_M}{z^N + \hat{a}_1 z^{N-1} + \hat{a}_2 z^{N-2} + \dots + \hat{a}_N}$$

or

$$X(z) = \left(b_0 z^{N-M} \right) \frac{\prod_{m=1}^M (z - z_m)}{\prod_{k=1}^N (z - \rho_k)} = \left(b_0 \right) \frac{\prod_{m=1}^M (1 - z_m z^{-1})}{\prod_{k=1}^N (1 - \rho_k z^{-1})}$$

- $z_m, 1 \leq m \leq M$, is called a zero of $X(z)$ since $X(z_m) = 0$.
- $\rho_k, 1 \leq k \leq N$, is called a pole of $X(z)$ since $X(\rho_k) = \infty$.
- Any Rational Z-Transform like $X(z)$ has always the same number of Zeros and poles;
 - If $N > M$, then $N-M$ Zeros are at origin (at zero) in Z-plane.
 - If $N < M$, then $M-N$ poles are at origin (at zero) in Z-plane.

Example 1:

$$x(n) = a^n u(n) \leftrightarrow X(z) = \frac{1}{1 - a z^{-1}} = \frac{z}{z - a},$$

$$|z| > |a|$$

Example 2:

$$x(n) = \begin{cases} a^n & 0 \leq n \leq M-1 \\ 0 & \text{otherwise} \end{cases}$$

$$X(z) = \sum_{n=0}^{M-1} (a z^{-1})^n = 1 + a z^{-1} + a^2 z^{-2} + \dots + a^{M-1} z^{-(M-1)} \quad \text{(I)}$$

$$\text{or } X(z) = \frac{1 - (a z^{-1})^M}{1 - a z^{-1}} = \frac{z^M - a^M}{z^{M-1}(z - a)} \quad \text{(II)}$$

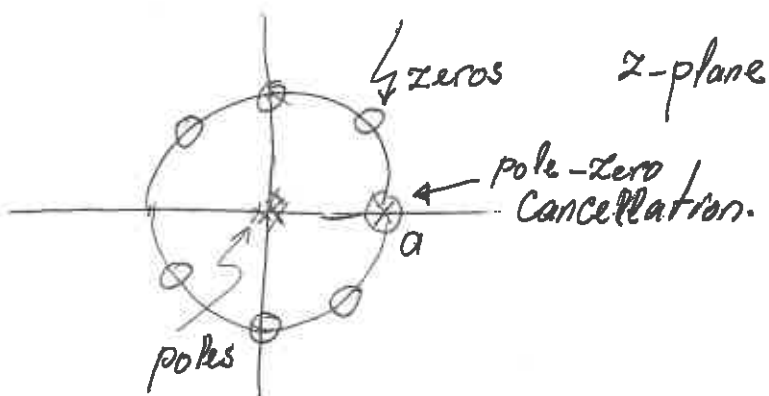
$$z^M - a^M = 0 \Rightarrow z^M = a^M e^{j2\pi k}, \quad k = 1, 2, 3, \dots, M-1$$

→ Same Number of poles and zeros of $X(z)$.

→ (I) can also be written as $\frac{z^{M-1} + a z^{M-2} + \dots + a^{M-1}}{z^{M-1}}$ $M-1$ poles at origin.

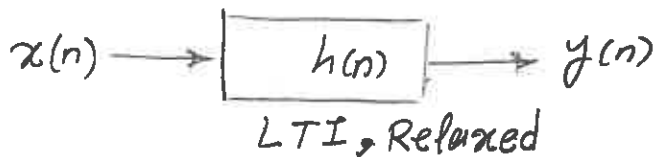
→ (II) has a pole and a zero at $z = a$ that cancel each other, and becomes like (I).

→ For $M=8 \Rightarrow$
(and $a > 0$)
(and real.)



Z-Transform Description of LTI System

For LTI Relaxed system of order $\max(N, M)$;



$$y(n) = x(n) * h(n) = - \sum_{k=1}^N a_k y(n-k) + \sum_{m=0}^M b_m x(n-m)$$

$$Y(z) = X(z) H(z)$$

where $H(z) = \sum_{n=-\infty}^{\infty} h(n) \bar{z}^{-n}$, $X(z) = \sum_{n=-\infty}^{\infty} x(n) \bar{z}^{-n}$

and $Y(z) = \sum_{n=-\infty}^{\infty} y(n) \bar{z}^{-n}$

→ For causal system: $H(z) = \sum_{n=0}^{\infty} h(n) \bar{z}^{-n}$

Definition: The Z-Transforms given above, such as

$$Y(z) = \sum_{n=-\infty}^{\infty} y(n) \bar{z}^{-n}, \text{ are called}$$

Two-sided Z-Transforms.

That is, all values of $y(n)$ for $-\infty \leq n \leq \infty$ are used.

- From $Y(z) = X(z)H(z)$ we get $H(z) = \frac{Y(z)}{X(z)}$, and

$$\left. \begin{aligned} |Y(z)| &= |X(z)| |H(z)| \\ \angle_{\text{phase}} Y(z) &= \angle X(z) + \angle H(z) \end{aligned} \right\} \text{ for any } z$$

and $ROC_Y = ROC_X \cap ROC_H$
(intersection)

- Taking Two-sided Z-Transform to the system's difference equation, we get

$$Y(z) = - \sum_{k=1}^N a_k \bar{z}^k Y(z) + \sum_{m=0}^M b_m \bar{z}^m X(z)$$

or

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{m=0}^M b_m \bar{z}^m}{1 + \sum_{k=1}^N a_k \bar{z}^k} = \sum_{n=0}^{\infty} h(n) \bar{z}^n$$

if system is causal.

Inversion of the Z-Transform

- Given $X(z)$ and ROC_x , three methods of finding $x(n)$ uniquely are:

$$x(n) \xleftrightarrow{Z} X(z), ROC_x$$

METHOD:

(1) Direct evaluation of

$$x(n) = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz$$

by contour integration.

(2) Expansion of $X(z)$ into a series of terms in variable z^+ and z^- .

(3) Partial-fraction expansion of $X(z)$ and Table lookup.

that is, decomposing $X(z)$ into sum of simple forms whose inverses can be found easily, and then using the superposition properties to find $x(n)$.

For example: $X(z) = X_1(z) + X_2(z) + \dots + X_L(z)$

where $ROC_x = \bigcap_{l=1}^L ROC_{x_l}$, $x_l(n) \xleftrightarrow{ROC_{x_l}} X_l(z)$

then $x(n) = \sum_{l=1}^L x_l(n)$.

- We consider only methods (2) and (3).

Inverse Z-Transform

Method (2): Power Series Expansion: $X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$.

Find $\left\{ \begin{array}{l} X(z) = \sum_{n=-\infty}^{\infty} C_n z^{-n} \text{ then } C_n = x(n). \\ \text{ROC}_X \end{array} \right.$

Example 1: • $X_1(z) = (1 - bz^{-1})^3 = 1 - 3bz^{-1} + 3b^2z^{-2} - b^3z^{-3}$

Thus $x_1(n) = \{ \underset{\substack{\uparrow \\ n=0}}{1} \quad -3b \quad +3b^2 \quad -b^3 \}$

$x_1(n)$ is causal since z^{-n} is for $n \leq 0$ (negative and zero powers of z)

• $X_2(z) = (a + bz)^2 = a^2 + 2abz^{+1} + b^2z^{+2}$

thus $x_2(n) = \{ b^2 \quad 2ab \quad \underset{\substack{\uparrow \\ n=0}}{a^2} \}$

is non-causal because powers of z are zero and positive values.

Example 2: $X(z)$ is a Rational function, $X(z) = \frac{P_M(z)}{D_N(z)}$.

Series expansion can be done by division, dividing $P_M(z)$ by $D_N(z)$ powers of z will be zero and/or negative, or positive depending on if $X(z)$ is causal ($ROC_X: |z| > |r_1|$), or Anti-causal ($ROC_X: |z| < |r_2|$), or non-causal ($ROC_X: |r_1| < |z| < |r_2|$), respectively.

- For $X(z) = \frac{B}{1 - a\bar{z}^{-1}}$, $ROC_X: |z| > |a|$ (causal)

Division:

$$\begin{array}{r|l}
 & 1 + a\bar{z}^{-1} + a^2\bar{z}^{-2} + a^3\bar{z}^{-3} + \dots + a^n\bar{z}^{-n} + \dots \\
 \hline
 1 - a\bar{z}^{-1} & \begin{array}{l} 1 \\ (1 - a\bar{z}^{-1}) - \\ + a\bar{z}^{-1} \\ \hline -(a\bar{z}^{-1} - a^2\bar{z}^{-2}) \\ \hline + a^2\bar{z}^{-2} \\ \hline -(a^2\bar{z}^{-2} - a^3\bar{z}^{-3}) \\ \hline + a^3\bar{z}^{-3} \\ \hline \vdots \end{array}
 \end{array}$$

Thus $x(n) = B a^n u(n)$
(causal).

- For $X(z) = \frac{C}{1 - a\bar{z}^{-1}} \quad |z| < |a| \text{ (Anti-causal)}$

Division:

$$\begin{array}{r}
 -\bar{a}^{-1}z^{-1} - \bar{a}^{-2}z^{-2} - \bar{a}^{-3}z^{-3} - \dots - \bar{a}^{-n}z^{-n} - \dots \\
 \hline
 -\bar{a}z^{-1} + 1 \quad \begin{array}{l} 1 - \\ -(1 - \bar{a}^{-1}z) \\ \hline +\bar{a}^{-1}z \\ -(\bar{a}^{-1}z - \bar{a}^{-2}z^2) \\ \hline +\bar{a}^{-2}z^2 \\ -(\bar{a}^{-2}z^2 - \bar{a}^{-3}z^3) \\ \hline +\bar{a}^{-3}z^3 \\ \vdots \end{array}
 \end{array}$$

Thus $x(n) = -C \bar{a}^{+n} U(-n-1)$
 (for $n \leq -1 \Rightarrow \text{Anti-causal}$)

→ Note: For anti-causal case numerator and denominator are written in reverse order of terms.

Example 3

$$X(z) = \frac{P_3(z)}{D_1(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3}}{1 - a z^{-1}} \quad |z| > |a| \quad (\text{Causal})$$

→ Note that $M=3$ and $N=1$ for $P_3(z)$ and $D_1(z)$

Step 1: We obtain division of $\frac{1}{1 - a z^{-1}}$ to obtain causal power series expansion similar to that in Example 2, that is

$$g(n) = a^n u(n).$$

Step 2: We can write

$$X(z) = b_0 \left(\frac{1}{1 - a z^{-1}} \right) + b_1 z^{-1} \left(\frac{1}{1 - a z^{-1}} \right) + b_2 z^{-2} \left(\frac{1}{1 - a z^{-1}} \right) + b_3 z^{-3} \left(\frac{1}{1 - a z^{-1}} \right)$$

and use superposition properties of z -Transform to write

$$x(n) = b_0 g(n) + b_1 g(n-1) + b_2 g(n-2) + b_3 g(n-3)$$

or

$$x(n) = b_0 a^n u(n) + b_1 a^{n-1} u(n-1) + b_2 a^{n-2} u(n-2) + b_3 a^{n-3} u(n-3).$$

Inverse Z-Transform

Method (3) : Partial Fraction Expansion of

$$X(z) = \frac{P_m(z)}{D_N(z)}, \text{ ROC}_X \quad r_1 < |z| < r_2$$

(Rational Function)

Let

$$X(z) = \frac{P_m(z)}{D_N(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \dots + b_m z^{-m}}{1 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3} + \dots + a_N z^{-N}}$$

Definition: If $N > m$, then $X(z)$ is called
Proper rational function.

Note: If $N \leq m$, then $X(z)$ can be written
 (by division) as a polynomial of order
 $m - N$ plus a proper rational function,

$$X(z) = c_0 + c_1 z^{-1} + c_2 z^{-2} + \dots + c_{m-N} z^{-(m-N)} + \frac{Q_{N-1}(z)}{D_N(z)}$$

Where $Y(z) = \frac{Q_{N-1}(z)}{D_N(z)}$ is proper rational function.

then $x(n) = g(n) + y(n)$ where

$$G(z) = \sum_{l=0}^{m-N} c_l z^{-l}.$$

• Let $X(z) = \frac{B_m(z)}{A_N(z)} = \frac{\sum_{m=0}^M b_m z^{-m}}{1 + \sum_{k=1}^N a_k z^{-k}}$, $N > M$
 with no pole-zero
 cancellation, be proper rational function. with ROC_x given,
 known in general as $r_1 < |z| < r_2$.

Problem: Find $x(n) \longleftrightarrow X(z)$, ROC_x .

Solution: There are many ways to find $x(n)$.
 We consider here the following approach.

Step 1: Write $X(z)$ in positive power of z by
 multiplying Numerator and Denominator by
 z^N . then

$$X(z) = \left(z^{N-M} \right) \frac{b_0 z^M + b_1 z^{M-1} + \dots + b_M}{z^N + a_1 z^{N-1} + \dots + a_N}$$

Step 2: Find poles of $X(z)$; p_k for $k=1, 2, \dots, N$.

$$\text{Thus } z^N + a_1 z^{N-1} + a_2 z^{N-2} + \dots + a_N = 0$$

$$\Rightarrow p_k \text{ for } 1 \leq k \leq N.$$

Case 1: poles are distinct, that is

$$p_i \neq p_j \quad 1 \leq i, j \leq N, \quad i \neq j$$

• Poles can be real
 or Complex.

Step 3: Write

$$G(z) = \frac{X(z)}{z} = \left(\frac{z^{N-M-1}}{z} \right) \frac{b_0 z^M + b_1 z^{M-1} + \dots + b_M}{(z-p_1)(z-p_2)\dots(z-p_N)} =$$

$$G(z) = \sum_{k=1}^N \frac{A_k}{(z-p_k)} \quad \text{--- (I)}$$

Constant A_k , $1 \leq k \leq N$ are then found as follows.

$$A_k = (z-p_k) G(z) \Big|_{z=p_k} \quad \text{for } k=1, 2, 3, \dots, N.$$

Step 4: For the known values of A_k and p_k , then

$$X(z) = z G(z) = \sum_{k=1}^N \frac{A_k z}{z-p_k}$$

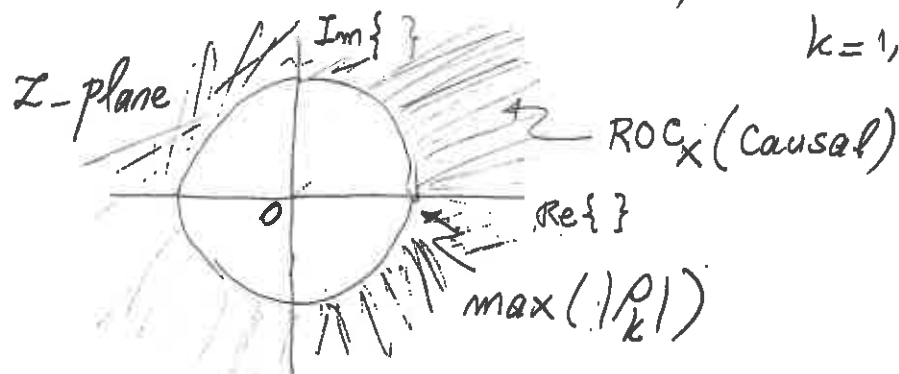
$$X(z) = \frac{A_1}{1-p_1 z^{-1}} + \frac{A_2}{1-p_2 z^{-1}} + \dots + \frac{A_k}{1-p_k z^{-1}} + \dots + \frac{A_N}{1-p_N z^{-1}}$$

Note 1: If $X(z)$ is causal every term in the (II) right side of (II) corresponds to a causal sequence. Thus

$$x(n) = A_1 (p_1)^n u(n) + A_2 (p_2)^n u(n) + \dots + A_k (p_k)^n u(n) +$$

where ROC for $\frac{A_k}{1-p_k z^{-1}}$, for every k , is $|p_k| < |z| < \infty$. (III)

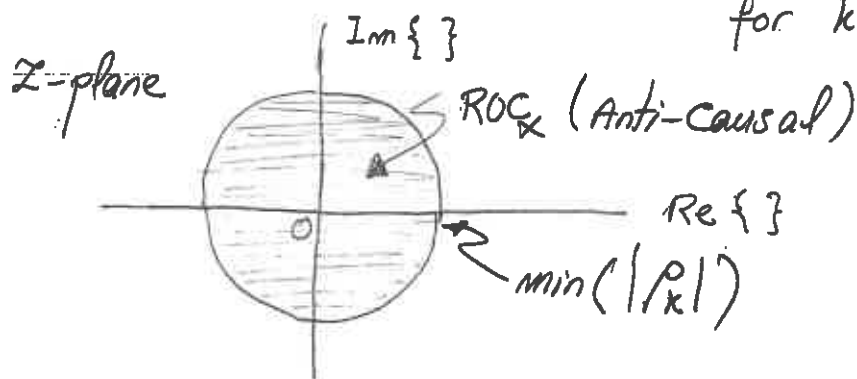
And, the ROC_x is $\max(|p_k|) < |z| < \infty$, for $k=1, 2, \dots, N$.



Note 2: If $X(z)$ is Anti-causal, then every term in right side of (II) corresponds to an Anti-causal sequence. Thus

$$x(n) = -A_1 (p_1)^n u(-n-1) - A_2 (p_2)^n u(-n-1) - \dots - A_N (p_N)^n u(-n-1).$$

And, the ROC_x is $0 \leq |z| < \min(|p_k|)$ (IV)
for $k=1, 2, 3, \dots, N$.



- The ROC for every $\frac{A_k}{1 - p_k z^{-1}}$ is $0 \leq |z| < |p_k|$.

Note 3: If $X(z)$ is Non-causal, then its ROC_X must be a ring, if it exists.

Thus we must assign some of the terms $\frac{A_k}{1 - p_k \bar{z}}$ to be causal and some Anti-causal in (II) so that intersection of ROCs of all terms will yield a ring.

→ For example;

Let $\frac{A_i}{1 - p_i \bar{z}}$ for $1 \leq i \leq N-L$ to have

$$ROC; |p_i| < |z| \leq \infty \text{ (causal)}$$

Let $\frac{A_j}{1 - p_j \bar{z}}$ for $N-L+1 \leq j \leq N$ to have

$$ROC; 0 \leq |z| < |p_j| \text{ (Anti-causal)}$$

So that ROC_X as a ring in z -plane will be:

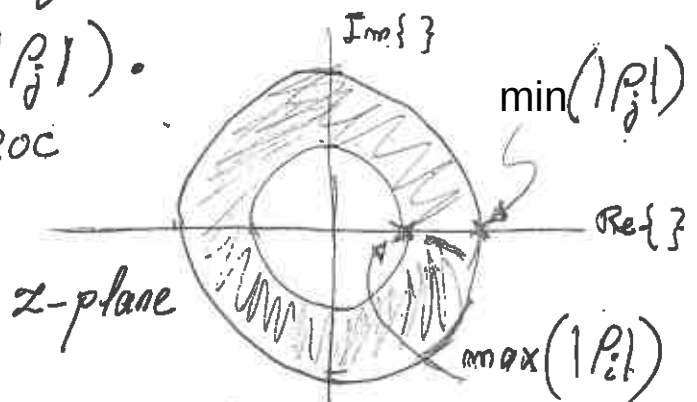
$$\max(|p_i|) < |z| < \min(|p_j|).$$

→ ROC_X is intersection of ROC of all terms in (II)

i.e.

$$ROC_X = \bigcap_k^N ROC_k$$

where ROC_k is for k th term in (II).



Thus,

$$x(n) = \sum_{i=1}^{N-L} A_i (\rho_i)^n u(n) - \sum_{j=N-L+1}^N A_j (\rho_j)^n u(n-1).$$

IMPORTANT NOTE: No pole can ever be in the region of Convergence (ROC).

→ Case 2: Poles of $X(z)$ are not distinct.

Suppose that pole ρ_1 is repeated r times, i.e. ρ_1 is of multiplicity r (i.e. $\rho_1 = \rho_2 = \dots = \rho_r$).

Then (I) in Step 3 is now written as

$$G(z) = \frac{A_1}{(z-\rho_1)} + \frac{A_2}{(z-\rho_1)^2} + \dots + \frac{A_r}{(z-\rho_1)^r} + \frac{A_{k+1}}{(z-\rho_{k+1})} + \dots + \frac{A_N}{(z-\rho_N)} \quad \text{(V)}$$

where $k = r+1, r+2, \dots, N$.

- The constants in numerators of (V) are calculated as follows.

$$\rightarrow A_k = (z-\rho_k) G(z) \Big|_{z=\rho_k} \text{ for } k=r+1, r+2, \dots, N$$

$$\rightarrow A_r = (z-\rho_1)^r G(z) \Big|_{z=\rho_1}$$

→ Now to find $A_1, A_2, \dots, A_{r-1}$, we equate both sides of (IV) by putting $r-1$ values for z that are not the same as ρ_1 and ρ_k for $r+1 \leq k \leq N$. This produces a set of $r-1$ linear equations in parameters $A_1, A_2, \dots, A_{r-1}$. Solving this set gives the values for constants $A_1, A_2, A_3, \dots, A_{r-1}$.

Thus we have

$$X(z) = z G(z) = \frac{A_1 z}{(z - \rho_1)} + \frac{A_2 z}{(z - \rho_1)^2} + \dots + \frac{A_r z}{(z - \rho_1)^r} + \frac{A_k z}{(z - \rho_k)} + \frac{A_{k+1} z}{(z - \rho_{k+1})} + \dots + \frac{A_N z}{(z - \rho_N)} \quad \text{(VI)}$$

Writing (VI) for negative powers of z yields

$$X(z) = \frac{A_1}{(1 - \rho_1^{-1} \bar{z}^{-1})} + \frac{A_2 \bar{z}^{-1}}{(1 - \rho_1^{-1} \bar{z}^{-1})^2} + \frac{A_3 \bar{z}^{-2}}{(1 - \rho_1^{-1} \bar{z}^{-1})^3} + \dots + \frac{A_r \bar{z}^{-(r-1)}}{(1 - \rho_1^{-1} \bar{z}^{-1})^r} + \frac{A_k}{(1 - \rho_k^{-1} \bar{z}^{-1})} + \dots + \frac{A_N}{(1 - \rho_N^{-1} \bar{z}^{-1})} \quad \text{(VII)}$$

Next step:

- Now that (VII) is well-defined for $X(z)$ with repeated poles, then inverse of each term in right side of (VII) is found as causal or Anti-causal depending on whether $X(z)$ is causal, Anti-causal, or Non-causal, as it was done for the case 1 (i.e. for distinct poles of $X(z)$).
- For Causal $X(z)$, ROC_X is $\max(|p_l|) < |z| \leq \infty$ for $l=1, 2, 3, \dots, N$.
- For Anti-causal $X(z)$, ROC_X is $0 \leq |z| < \min(|p_l|)$ for $l=1, 2, 3, \dots, N$.
- For Non-causal $X(z)$, ROC_X is a ring in z -plane specified by
for example: $\max_{\alpha} (|p_{\alpha}|) < |z| < \min |p_{\beta}|$
where p_{β} is for the Anti-causal terms and p_{α} is for the causal terms in $X(z)$ of (VII).
That is, $ROC_X = \bigcap_{k=1}^N ROC_k$, where ROC_k is that of every term in (VII), i.e. the intersection of ROC of all terms in (VII).

Exercise: show the following:

- For ROC $|z| > |a|$ (causal):

$$* \quad u(n) \longleftrightarrow U(z) = \frac{1}{1-\bar{z}^{-1}} = \frac{z}{z-1}$$

$$* \quad a^{n-1} u(n-1) \longleftrightarrow \frac{\bar{z}^{-1}}{1-a\bar{z}^{-1}} = \frac{1}{z-a}$$

$$* \quad (n-1)a^{n-2} u(n-1) \longleftrightarrow \frac{\bar{z}^{-2}}{(1-a\bar{z}^{-1})^2} = \frac{1}{(z-a)^2}$$

$$* \quad \left(\frac{1}{2!}\right)(n-1)(n-2)a^{n-3} u(n-1) \longleftrightarrow \frac{\bar{z}^{-3}}{(1-a\bar{z}^{-1})^3} = \frac{1}{(z-a)^3}$$

$$* \quad na^n u(n) \longleftrightarrow \frac{a\bar{z}^{-1}}{(1-a\bar{z}^{-1})^2} = \frac{az}{(z-a)^2}$$

- For ROC $|z| < |a|$ (Anti-causal):

$$* \quad -a^{n-1} u(-n) \longleftrightarrow \frac{\bar{z}^{-1}}{1-a\bar{z}^{-1}} = \frac{1}{z-a}$$

$$* \quad -(n-1)a^{n-2} u(-n) \longleftrightarrow \frac{\bar{z}^{-2}}{(1-a\bar{z}^{-1})^2} = \frac{1}{(z-a)^2}$$

$$* \quad \left(\frac{1}{2!}\right)(n-1)(n-2)a^{n-3} u(-n) \longleftrightarrow \frac{\bar{z}^{-3}}{(1-a\bar{z}^{-1})^3} =$$

$$= \frac{1}{(z-a)^3}$$

• Examples For Inverse Z-Transform:

Example 1: Find $x(n)$ as inverse of

$$X(z) = \frac{1}{1 + (\frac{5}{6})\bar{z}^1 - \frac{1}{6}\bar{z}^2}$$

Solution:

$$X(z) = \frac{1}{1 + \frac{5}{6}\bar{z}^1 - \frac{1}{6}\bar{z}^2} = \frac{z^2}{z^2 + \frac{5}{6}z - \frac{1}{6}} = \frac{z^2}{(z - \frac{1}{2})(z - \frac{1}{3})}$$

$$G(z) = \frac{X(z)}{z} = \frac{z}{(z - \frac{1}{2})(z - \frac{1}{3})} = \frac{A_1}{(z - \frac{1}{2})} + \frac{A_2}{(z - \frac{1}{3})}$$

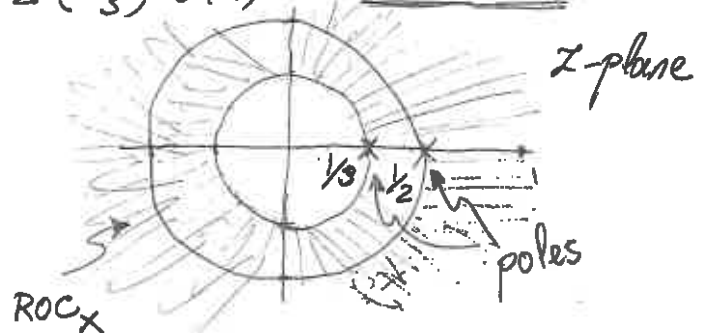
$$A_1 = \left. \frac{z}{z - \frac{1}{3}} \right|_{z = \frac{1}{2}} = \frac{\frac{1}{2}}{\frac{1}{2} - \frac{1}{3}} = 3$$

$$A_2 = \left. \frac{z}{z - \frac{1}{2}} \right|_{z = \frac{1}{3}} = \frac{\frac{1}{3}}{\frac{1}{3} - \frac{1}{2}} = \frac{\frac{1}{3}}{-\frac{1}{6}} = -2$$

$$X(z) = 3 \frac{z}{z - \frac{1}{2}} - 2 \frac{z}{z - \frac{1}{3}} = 3 \frac{1}{1 - \frac{1}{2}\bar{z}} - 2 \frac{1}{1 - \frac{1}{3}\bar{z}}$$

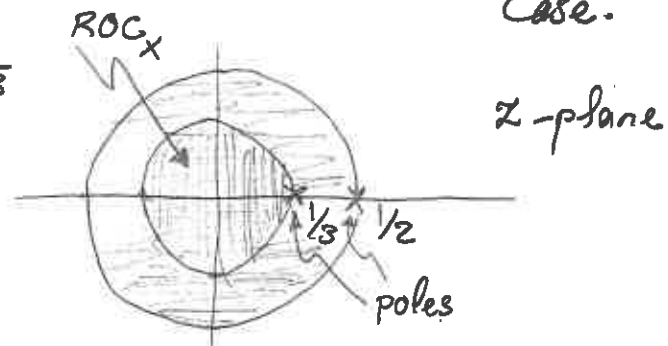
• $x(n) = 3\left(\frac{1}{2}\right)^n u(n) - 2\left(\frac{1}{3}\right)^n u(n)$ Causal case

Thus $ROC_x: \frac{1}{2} < |z| \leq \infty$



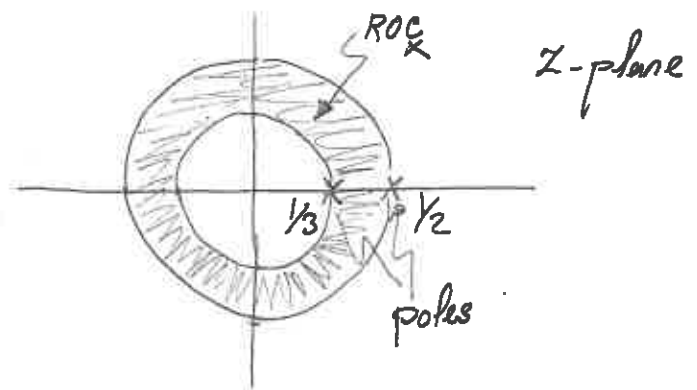
- $x(n) = -3\left(\frac{1}{2}\right)^n u(-n-1) + 2\left(\frac{1}{3}\right)^n u(-n-1)$ Anti-causal Case.

Thus $ROC_x: 0 \leq |z| < \frac{1}{3}$



- $x(n) = -3\left(\frac{1}{2}\right)^n u(-n-1) - 2\left(\frac{1}{3}\right)^n u(n)$ Non-causal Case.

Thus $ROC_x: \frac{1}{3} < |z| < \frac{1}{2}$



Example 2: Find $x(n)$ as inverse of

$$X(z) = \frac{1}{(1+\bar{z}^{-1})(1-2\bar{z}^{-1}+\bar{z}^{-2})}$$

Solution:

$$X(z) = \frac{z^3}{(z+1)(z^2-2z+1)} = \frac{z^3}{(z+1)(z-1)^2}$$

$$G(z) = \frac{X(z)}{z} = \frac{z^2}{(z+1)(z-1)^2} = \frac{A}{(z+1)} + \frac{B_1}{(z-1)} + \frac{B_2}{(z-1)^2}$$

$$A = \frac{z^2}{(z-1)^2} \Big|_{z=-1} = \frac{1}{4} \quad \underline{\underline{(I)}}$$

$$B_2 = \frac{z^2}{z+1} \Big|_{z=1} = \frac{1}{2}$$

Let $z=0$ on both sides of (I). Then

$$A - B_1 + B_2 = 0 \Rightarrow B_1 = \frac{3}{4}$$

Thus $X(z) = z G(z) = \left(\frac{1}{4}\right) \frac{z}{(z+1)} + \left(\frac{3}{4}\right) \frac{z}{(z-1)} + \left(\frac{1}{2}\right) \frac{z}{(z-1)^2}$

or

$$X(z) = \left(\frac{1}{4}\right) \frac{1}{(1+\bar{z}^{-1})} + \left(\frac{3}{4}\right) \frac{1}{(1-\bar{z}^{-1})} + \left(\frac{1}{2}\right) \frac{\bar{z}^{-1}}{(1-\bar{z}^{-1})^2}$$

→ • For $X(z)$ causal:

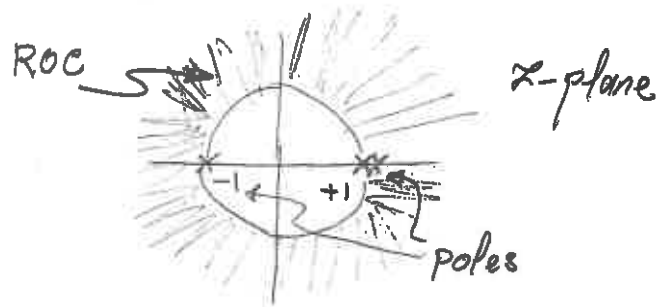
$$x(n) = \left(\frac{1}{4}\right) (-1)^n u(n) + \left(\frac{3}{4}\right) (1)^n u(n) + \left(\frac{1}{2}\right) n (1)^n u(n)$$

since $n u(n) \xleftrightarrow{Z} \frac{\bar{z}^{-1}}{(1-\bar{z}^{-1})^2}$.

And

$$ROC_x: 1 < |z| \leq \infty$$

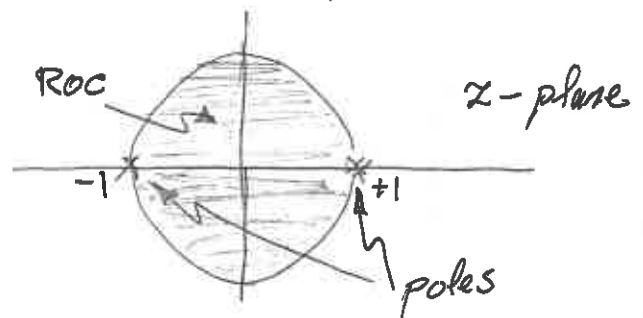
(Causal)



→ • For $X(z)$ Anti-causal:

$$x(n) = -\left(\frac{1}{4}\right)(-1)^n u(-n-1) - \left(\frac{3}{4}\right)(1)^n u(-n-1) - \left(\frac{1}{2}\right)(n)(1)^n u(-n-1)$$

$$ROC_x: 0 \leq |z| < 1$$



→ • For $X(z)$ Non-causal:

$X(z)$ cannot be Non-causal because No ring exists in z -plane for ROC_x as $r_1 \neq |z| \neq r_2$, i.e. for $|z| < 1$!

One-Sided Z-Transform:

Definition: For a given $x(n)$,

- Two-sided Z-Transform is $X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$.

Thus all values of $x(n)$ are needed for $-\infty \leq n \leq +\infty$ to find $X(z)$.

- One-sided Z-Transform of $x(n)$ is

$$X^+(z) = \sum_{n=0}^{\infty} x(n) z^{-n}.$$

Thus all values of $x(n)$ are only needed for $0 \leq n \leq +\infty$.

Notes:

- $X^+(z)$ does not contain information about the signal $x(n)$ for $n < 0$.
- $X^+(z)$ is unique only for causal $x(n)$.
- $X^+(z) = X(z)$ for $x(n)u(n)$. Thus

ROC_{x^+} is always exterior of a circle,

i.e. $ROC_{x^+} : r_1 < |z| \leq +\infty$, where

$r_1 = \max(|\rho_k|)$, ρ_k are poles of $X^+(z)$.

Example 1:

Find $X^+(z)$ for the following $x(n)$:

(1) $x_1(n) = \{1, 2, 5, 7, 8, 12\} = 6(n+2) + 26(n+1) +$
 $\quad \quad \quad \uparrow$
 $\quad \quad \quad n=0$ $\quad \quad \quad + 56(n) + 76(n-1) +$
 $\quad \quad \quad + 86(n-2) + 126(n-3)$

$$X_1^+(z) = \sum_{n=0}^{\infty} x_1(n) z^{-n} = \sum_{n=0}^3 x_1(n) z^{-n}$$

$$X_1^+(z) = 5 + 7z^{-1} + 8z^{-2} + 12z^{-3} \quad \text{(I)}$$

But

$$X_1(z) = \sum_{n=-\infty}^{\infty} x_1(n) z^{-n} = \sum_{n=-2}^3 x_1(n) z^{-n}$$

$$X_1(z) = z^{+2} + 2z^{+1} + \underbrace{\{5 + 7z^{-1} + 8z^{-2} + 12z^{-3}\}}_{X_1^+(z)} \quad \text{(II)}$$

(2) $v_1(n) = x_1(n-2) = \{1, 2, 5, 7, 8, 12\}$
 $\quad \quad \quad \uparrow$
 $\quad \quad \quad n=0$

Now

$$V_1^+(z) = \sum_{n=0}^5 v_1(n) z^{-n} = x_1(-2) + x_1(-1) z^{-1} +$$

$$+ x_1(0) z^{-2} + x_1(1) z^{-3} + x_1(2) z^{-4} +$$

$$+ x_1(3) z^{-5}$$

or

$$V_1^+(z) = z^{-2} \left\{ [x_1(-2) z^{+2} + x_1(-1) z^{+1}] + \right.$$

$$\left. + [x_1(0) + x_1(1) z^{-1} + x_1(2) z^{-2} + x_1(3) z^{-3}] \right\}$$

$$\quad \quad \quad \underbrace{\hspace{10em}}_{X_1^+(z)}$$

Thus

$$\begin{aligned}
 V_1^+(z) &= \mathcal{Z}^+ \{x(n-2)\} = \\
 &= z^{-2} \left\{ \sum_{l=-1}^{-2} x_1(l) z^{-l} + X_1^+(z) \right\} \\
 &= z^{-2} \left\{ \sum_{l=1}^2 x_1(-l) z^{+l} + X_1^+(z) \right\}
 \end{aligned}$$

So, in general:

→ If $x(n) \longleftrightarrow X^+(z)$, then

$$v(n) = x(n-k) \longleftrightarrow V^+(z) = z^{-k} \left\{ \sum_{l=1}^k x(-l) z^{+l} + X^+(z) \right\} \quad \text{(I)}$$

* In similar way we have:

→ If $x(n) \longleftrightarrow X^+(z)$, then

$$g(n) = x(n+k) \longleftrightarrow G^+(z) = z^{+k} \left\{ X^+(z) - \sum_{l=0}^{k-1} x(l) z^{-l} \right\} \quad \text{(II)}$$

Note that $x(-l)$ for $l=1, 2, \dots, k$ are considered as initial values of $x(n)$ in (I) for finding $V^+(z)$ or One-sided $\mathcal{Z}$ -Transform of $x(n)$.

Note: properties of one-sided and two-sided Z-Transforms of $x(n)$ are very similar except for shifting of $x(n)$ by $\pm k$ as shown by (I) and (II) above.

- Final value Theorem:

$$\text{If } x(n) \xleftrightarrow{\mathcal{Z}^+} X^+(z)$$

$$\text{then } \lim_{n \rightarrow \infty} x(n) = \lim_{z \rightarrow 1} [(z-1) X^+(z)]$$

where this limit exists if ROC of $(z-1) X^+(z)$ includes unit circle, i.e. it includes $|z|=1$ in z-plane.

→ This Theorem can be used to see asymptotic behavior of signal $x(n)$ once its Z-Transform is known but not the signal $x(n)$ itself.

Asymptotic behavior of $x(n)$ shows the steady state value of $x(n)$.

Using Z-Transform To Solve Difference Equation

- Using the Z-Transform to easily solve difference equation

$$y(n) = - \sum_{k=1}^N a_k y(n-k) + \sum_{m=0}^M b_m x(n-m) \quad (\underline{\text{I}})$$

for a given input $x(n)$ and initial conditions $y(-1), y(-2), \dots, y(-N)$, and for $y(n), n \geq 0$.

The order of $(\underline{\text{I}})$ is $\max(N, M)$.

We also assume $x(n)$ is causal ($x(n)=0$ for $n < 0$).

- We must use One-sided Z-Transform to solve $(\underline{\text{I}})$.
- If the system $(\underline{\text{I}})$ is at rest or relaxed then all initial condition values are zero. In this case One-sided and Two-sided Z-Transforms are the same and can be used to solve $(\underline{\text{I}})$.

Otherwise, the one-sided Z-Transform must be used with its shifted properties which were discussed in order to account for the effects of non-zero initial condition values.

Example 1: Find $y(n)$ for $n \geq 0$ for a given $x(n)$ where the causal system is given by

$$y(n) = a y(n-1) + x(n), \quad -1 < a < 1$$

and $y(-1) \neq 0$.

Solution: Using One-sided Z-Transform we get

$$Y^+(z) = a \bar{z}^{-1} \{ Y^+(z) + y(-1)z \} + X^+(z)$$

$$Y^+(z) = a \bar{z}^{-1} Y^+(z) + a y(-1) + X^+(z)$$

and

$$Y^+(z) = \underbrace{\frac{a y(-1)}{1 - a \bar{z}^{-1}}}_{\text{Zero-Input response}} + \underbrace{\frac{X^+(z)}{1 - a \bar{z}^{-1}}}_{\text{Zero-state response}}$$

Note: Zero-state response is

$$x(n) * h(n) \longleftrightarrow X^+(z) \cdot H(z)$$

thus $H(z) = \frac{1}{1 - a \bar{z}^{-1}} \longleftrightarrow h(n) = a^n u(n)$

- If $x(n) = b^n u(n)$ for $|b| < 1$, then

$$X^+(z) = \frac{1}{1 - b \bar{z}^{-1}}$$

Thus

$$Y^+(z) = \frac{ay(-1)}{1-a\bar{z}^{-1}} + \frac{1}{(1-a\bar{z}^{-1})(1-b\bar{z}^{-1})}$$

Now let

$$P(z) = \frac{1}{(1-a\bar{z}^{-1})(1-b\bar{z}^{-1})} = \frac{z^2}{(z-a)(z-b)}$$

and $G(z) = \frac{P(z)}{z} = \frac{z}{(z-a)(z-b)} = \frac{A}{z-a} + \frac{B}{z-b}$

then

$$A = \frac{z}{z-b} \Big|_{z=a} = \frac{a}{a-b}$$

$$B = \frac{z}{z-a} \Big|_{z=b} = \frac{-b}{a-b}$$

and

$$Y^+(z) = ay(-1) \frac{1}{(1-a\bar{z}^{-1})} + \left(\frac{a}{a-b}\right) \frac{1}{1-a\bar{z}^{-1}} - \left(\frac{b}{a-b}\right) \frac{1}{1-b\bar{z}^{-1}}$$

and solution is:

$$y(n) = \underbrace{y(-1) a^{n+1} u(n)}_{\text{zero-input response}} + \underbrace{\left(\frac{1}{a-b}\right) \left\{ a^{n+1} u(n) - b^{n+1} u(n) \right\}}_{\text{zero-state response}}$$

Note

If $n \rightarrow \infty$ then both responses go to zero since $-1 < |a|, |b| < 1$. That is, both responses are transient. Then $y(n) \rightarrow 0$ for $n \rightarrow \infty$. Thus the steady state response of the system is zero.

- If $x(n) = u(n)$, i.e. $b=1$, then

$$y(n) = \underbrace{y(-1)a^{n+1}u(n) + \frac{a^{n+1}}{a-1}u(n)}_{\text{Transient response}} - \underbrace{\frac{1}{a-1}u(n)}_{\text{Steady state response}}$$

Note: If $n \rightarrow \infty$ then transient response goes to zero while the steady state response remain as

$$y_{ss}(n) = + \left(\frac{1}{1-a} \right) u(n)$$

for as long as the input $x(n) = u(n)$ exists.

Note: If the initial condition was zero, i.e. $y(-1) = 0$, then applying Two-sided Z-Transform to $y(n) = ay(n-1) + x(n)$, $-1 < |a| < 1$ would have yielded;

$$y(n) = \left(\frac{1}{a-b} \right) \left\{ a^{n+1}u(n) - b^{n+1}u(n) \right\} \text{ for } x(n) = b^n u(n),$$

$-1 < |b| < 1$, and

$$y(n) = \left(\frac{1}{a-1} \right) \left\{ a^{n+1}u(n) - u(n) \right\} \text{ for } x(n) = u(n).$$

That is, $y(n) = h(n) * u(n)$.

Example 2: Find impulse response $h(n)$ of the causal system

$$y(n) = -ay(n-1) - by(n-2) + cx(n-1)$$

for $x(n) = \left(\frac{1}{4}\right)^n u(n-3)$ and $y(-1) = 4$,

$$y(-2) = -3.$$

Solution

To find impulse response $h(n) \leftrightarrow H(z)$ where $h(n) = 0$ for $n < 0$, we must let or assume, that

(1) $x(n) = \delta(n)$ and (2) Zero-initial Conditions.

We can apply two-sided (or equivalently the one-sided) Z-Transform to the difference equation.

Thus

$$Y(z) = -a\bar{z}^{-1}Y(z) - b\bar{z}^{-2}Y(z) + c\bar{z}^{-1}X(z)$$

$$\frac{Y(z)}{X(z)} = H(z) = \frac{c\bar{z}^{-1}}{1 + a\bar{z}^{-1} + b\bar{z}^{-2}}$$

where $\delta(n) \leftrightarrow X(z) = 1$.

$$\text{Thus } Y(z) = H(z) = \frac{c\bar{z}^{-1}}{1 + a\bar{z}^{-1} + b\bar{z}^{-2}} = \frac{cx}{z^2 + az + b}$$

Let $z^2 + az + b = (z - \alpha)(z - \beta)$

Then by partial fraction expansion we get

$$Y(z) = H(z) = \left(\frac{c}{\alpha - \beta} \right) \left\{ \frac{1}{1 - \alpha z^{-1}} - \frac{1}{1 - \beta z^{-1}} \right\}$$

$$\text{and } h(n) = \left(\frac{c}{\alpha - \beta} \right) \{ \alpha^n u(n) - \beta^n u(n) \}$$

Note that the poles α and β are the characteristic values of the difference equation (as discussed in Chapter 2).

Causality and Stability In z-Domain

- A causal LTI system has impulse response

$$h(n) = 0 \text{ for } n < 0$$

and $H(z)$ has ROC that is outside a disk of radius r , i.e. $r < |z| \leq \infty$.

This is necessary and sufficient condition for the LTI system to be causal.

- A LTI system is BIBO stable if and only if (iff)

$$\sum_{n=-\infty}^{\infty} |h(n)| \leq M_h < \infty.$$

Now $H(z) = \sum_{n=-\infty}^{\infty} h(n) z^{-n}$ and

$$|H(z)| < \infty \text{ for any } z \text{ in the } \text{ROC}_H.$$

$$\text{But } |H(z)| = \left| \sum_{n=-\infty}^{\infty} h(n) z^{-n} \right| \leq \sum_{n=-\infty}^{\infty} |h(n)| |z|^{-n}$$

On the unit circle in z -plane, i.e. for $|z|=1$, we then have

$$|H(z)| \Big|_{|z|=1} \leq \sum_{n=-\infty}^{\infty} |h(n)| < \infty.$$

Thus $H(z)$ represent a BIBO stable LTI system iff ROC_H contains unit circle, or

or $ROC_H: r_1 < |z| < r_2$ where $r_1 < 1 < r_2$.

→ • From the iff conditions for causality and stability of a LTI system with system transfer function $H(z)$ we conclude that

→ A LTI causal system is BIBO stable

iff ROC_H for $H(z)$ contains unit circle. That is, iff ROC_H is $1 < |z| \leq \infty$.

$|z|=1 \Rightarrow$ unit circle.

→ Now for a causal LTI system, the poles of $H(z)$ are p_k for $k=1, 2, 3, \dots, N$, where N is the order of system.

then ROC_H is $\max(|p_k|) < |z| \leq \infty$.

→ Therefore, a causal LTI system is BIBO stable iff all poles of $H(z)$ are inside unit circle, i.e. $|p_k| < 1$ for $k=1, 2, \dots, N$.

Note An Anti-causal or a Non-causal LTI system is stable iff their ROC_H contain unit circle.

Example:

A LTI system is given as

$$H(z) = \frac{3 - 4z^{-1}}{1 - 3.5z^{-1} + 1.5z^{-2}}$$

specify the ROC_H and find $h(n) \longleftrightarrow H(z)$ for the following cases:

- (a) The system is stable
- (b) The system is causal
- (c) The system is Anti-causal

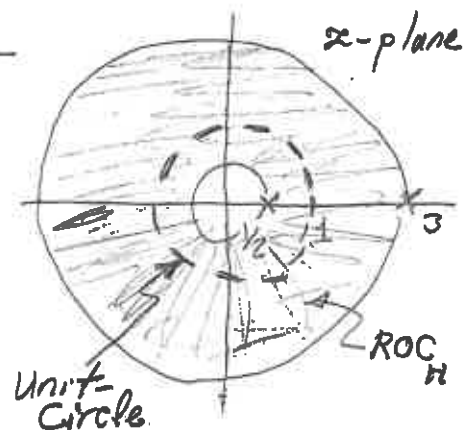
Solution:

(a) By partial fraction expansion we get

$$H(z) = \frac{1}{1 - \frac{1}{2}z^{-1}} + \frac{2}{1 - 3z^{-1}}$$

Since system is stable then

ROC_H must be $\frac{1}{2} < |z| < 3$ to include unit circle, and system is Non-causal.



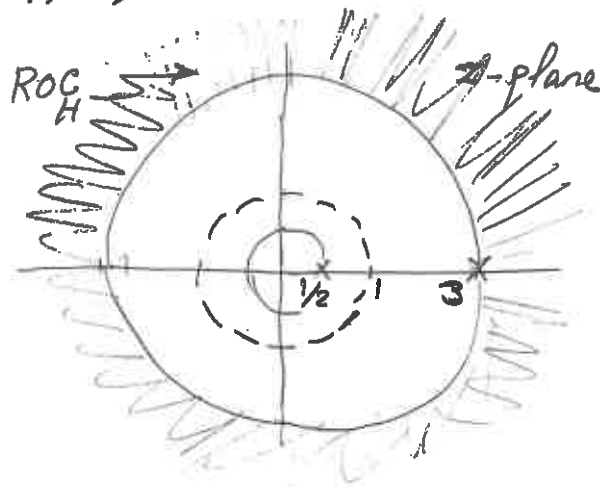
Thus we must have

$$h(n) = \left(\frac{1}{2}\right)^n u(n) - 2 \left(\frac{1}{3}\right)^n u(-n-1)$$

Note: $h(n) \rightarrow 0$ for $n \rightarrow +\infty$
and $h(n) \rightarrow 0$ for $n \rightarrow -\infty$ } i.e. stable

(b) Since $H(z)$ must be causal, then

ROC_H is $3 < |z| \leq \infty$
and it does not include
unit circle, thus it is
unstable.

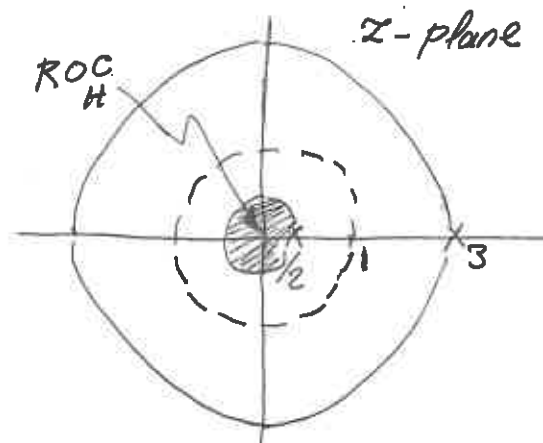


$$h(n) = \left(\frac{1}{2}\right)^n u(n) + 2 \left(\frac{1}{3}\right)^n u(n).$$

$h(n) \rightarrow \infty$ if $n \rightarrow \infty$. } unstable.

(c) Since $H(z)$ must be anti-causal, then

ROC_H is $0 \leq |z| < \frac{1}{2}$
and it does not include
unit circle, thus it is
unstable.



$$h(n) = -\left(\frac{1}{2}\right)^n u(-n-1) - 2 \left(\frac{1}{3}\right)^n u(-n-1)$$

$h(n) \rightarrow \infty$ if $n \rightarrow -\infty$. } unstable.

Exercise:Problem:

Consider the LTI system given by

$$y(n) = -\frac{4}{3} y(n-1) + \frac{4}{3} y(n-2) + x(n-1)$$

where $y(-1)=0$, $y(-2)=2$, $x(n)=u(n)$.

Using the Z-Transform,

- (a) find impulse response $h(n)$ of the system to be stable.
- (b) find the ROC_y of $Y(z)$.
- (c) find $y(n)$, for $n \geq 0$, for the stable system.

Schur - Cohn Stability Test

- A LTI Causal system given by its system Transfer function $H(z)$, where

$$h(n) \longleftrightarrow H(z) = \frac{Q_M(z)}{A_N(z)} = \frac{Q_M(z)}{1 + \sum_{k=1}^N a_k z^{-k}}$$

is stable iff all poles of $H(z)$ are inside unit circle, i.e. the roots of

$$A_N(z) = 1 + \sum_{k=1}^N a_k z^{-k} \text{ all have magnitude less than } 1.$$

- Thus we do not need to find every root of A_N explicitly, but just to find if magnitude of every root is or is not less than 1 to evaluate stability of the LTI Causal system.
- The Schur-Cohn determines if every root of a polynomial, like $A_N(z)$ has magnitude less than one, or if there is root of $A_N(z)$ with magnitude not less than one.

Thus the Schur-Cohn method can be used to test stability of a LTI causal system.

Schur-Cohn Stability Test Algorithm

For $H(z) = \frac{Q_m(z)}{A_N(z)}$
 $A_N(z) = 1 + \sum_{k=1}^N a_k z^{-k}$
 a LTI causal system

Step 1: Initialization:

Define $A_m(z) = \sum_{k=0}^m a_m(k) z^{-k}$, $a_m(0) = 1$

Thus $A_N(z) = A_m(z)$ for $m = N > 0$.

Let $m = N$, go to step 2.

Step 2:

Let $a_m(m) = K_m$.

- If $|K_m| > 1$, Then stop. $H(z)$ is not stable because there is at least a pole of $H(z)$, ρ_k , (a root of $A_N(z)$) that is not inside unit circle, i.e. $|\rho_k| \geq 1$.
- If $|K_m| < 1$, go to step 3.

Step 3:

- If $m=0$, then stop. $H(z)$ is stable since all poles of $H(z)$, (all roots of $A_N(z)$), are inside unit circle. And, $A_0(z) = B_0(z) = 1$.
- If $m > 0$, then for $A_m(z) = \sum_{k=0}^m a_m(k) z^{-k}$, where $a_m(0) = 1$.

Find: $B_m(z) = z^{-m} A_m(z^{-1}) = \sum_{k=0}^m a_m(m-k) z^{-k}$.

Note coefficients of $B_m(z)$ are the same as those of $A_m(z)$ but in reverse order.
Go to step 4.

Step 4:

- Find $A_{m-1}(z) = \sum_{k=0}^{m-1} a_{m-1}(k) z^{-k}$ from

$$A_{m-1}(z) = \frac{A_m(z) - K_m B_m(z)}{1 - (K_m)^2}$$

- Let $m \leftarrow m-1$,
- If $m=0$, go to step 3.
- If $m > 0$ go to step 2.

Note: • All roots of $A_N(z)$ are inside unit circle and $H(z)$ is stable iff

$$|K_m| < 1, \text{ for } m = N, N-1, \dots, 1.$$

- If any $|K_m| > 1$ for $1 \leq m \leq N$, then $H(z)$ has at least a pole not inside of unit circle, Thus the LTI causal system is not stable.
- If any $|K_m| = 0$, then the algorithm changes since $K_m = a_m(m)$ and order m of $A_m(z) = \sum_{k=0}^m a_m(k) z^{-k}$ reduces by one. That is $A_{m-1}(z) \equiv A_m(z)$ for $K_m = 0$.
- If $|K_m| = 1$, the algorithm is stopped and $H(z)$ is unstable since there may be poles outside or on the unit circle.

Example: $H(z)$ of a LTI causal system is given by

$$H(z) = \frac{7 - 5z^{-1}}{1 - (\frac{1}{12})z^{-1} - (\frac{11}{24})z^{-2} + (\frac{1}{8})z^{-3}}$$

Find if $H(z)$ is stable or not.

Solution

$$A_3(z) = 1 - (\frac{1}{12})z^{-1} - (\frac{11}{24})z^{-2} + (\frac{1}{8})z^{-3}$$

Thus $|K_3| = |\frac{1}{8}| = 0.125 < 1 \quad \checkmark$

Now, $B_3(z) = \frac{1}{8} - (\frac{11}{24})z^{-1} - (\frac{1}{12})z^{-2} + z^{-3}$

$$A_2(z) = \frac{A_3(z) - K_3 B_3(z)}{1 - K_3^2}$$

$$A_2(z) = 1 - (\frac{5}{189})z^{-1} - (\frac{86}{189})z^{-2}$$

Thus $|K_2| = \frac{86}{189} = 0.455026455 < 1 \quad \checkmark$

Now

$$B_2(z) = -(\frac{86}{189}) - (\frac{5}{189})z^{-1} + z^{-2}$$

and

$$A_1(z) = \frac{A_2(z) - K_2 B_2(z)}{1 - (K_2)^2} = 1 - 0.0305229z^{-1}$$

$$|K_1| = 0.0305229 < 1 \quad \checkmark$$

Thus $H(z)$ is stable since $|K_m| < 1$ for $m = 3, 2, 1$.

The actual poles of $H(z)$ are

$$p_1 = \frac{1}{2}, p_2 = \frac{1}{3}, p_3 = -\frac{3}{4}.$$

Exercise: using Schur-Cohn method
evaluate stability of a LTI
Causal system given by $H(z)$
as follows.

$$(1) H(z) = \frac{1}{1 - (\frac{7}{4})z^{-1} - (\frac{1}{2})z^{-2}}$$

$$(2) H(z) = \frac{3 - 5z^{-2}}{4 - \frac{10}{3}z^{-1} + \frac{2}{3}z^{-2}}$$